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Entropy of the Generalized Coleman-Hepp Model

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Relaxation and decoherence properties of the exactly solvable Coleman-Hepp model are studied by determining the time evolution of the entropy. With the aid of our previous results, we can calculate the entropy by solving an eigenvalue problem for the reduced density matrices. Detailed numerical studies are performed to clarify the decoherence processes.

KEYWORDS: entropy, Coleman-Hepp model, relaxation, decoherence, spin coherent state

§1. Introduction

In 1972, Hepp explicitly formulated a model based on work by Coleman.[1] He discussed the measurement problem of quantum mechanics. The model is a simple one-dimensional system composed of an incoming particle and an array of N spins with $S = 1/2$, S being the magnitude of a spin. Subsequently, a generalized version of the Coleman-Hepp model[2,3] was introduced, which takes into account the energy exchange effect and allows arbitrary spin magnitude S.

In previous studies,[4-6] we examined the relaxation and the decoherence processes of the generalized Coleman-Hepp model by a method of spin coherent state representation.[7,8] Thus, for instance, we can clarify the decoherence process when S becomes large.

It is important to note that the model is one of a few examples which can be solved exactly to provide details of the relaxation processes. In contrast with conventional theories of relaxation and decoherence, where the interaction between the relevant system and a reservoir is assumed to be weak, with this model we can study relaxation behavior even for strong interaction. Namely, we are able to determine the density matrices, the averages of observables and the quasi-probability functions.

In this paper, we determine the time evolution of the quantum mechanical entropy as a measure of decoherence. In particular, the entropy is rigorously calculated to examine coherence properties of the spin state using the method of studying spin relaxation processes.[9]

§2. Short Summary of the Generalized Coleman-Hepp Model

The Coleman-Hepp model is composed of a spin one-half incident particle and an array of N spins $\{\boldsymbol{S}_l\}$ $(l = 1, 2, \cdots, N)$ called a detector. The magnitude of each detector spin $\{S_l\}$ is arbitrary. The incident particle moves into the detector with a constant velocity v and interacts with the detector spins. The total Hamiltonian of this model is given by

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{P}_+ \mathcal{H}_1, \tag{2.1}$$

where

$$\mathcal{H}_0 = \mathcal{H}_I + \mathcal{H}_D, \tag{2.2}$$

$$\mathcal{H}_I = vP + \hbar\omega_I I^z, \tag{2.3}$$

$$\mathcal{H}_D = \sum_{l=1}^{N} \hbar\omega_l S_l^z. \tag{2.4}$$

The quantities vP and $\hbar\omega_I$ are the kinetic energy and the Zeemann energy, respectively, I^z being the z-component of the spin, while $\hbar\omega_l$ is the l-th spin energy of the detector.

The interaction Hamiltonian $\mathcal{H}_1$ is written as follows:

$$\mathcal{H}_1 = \frac{1}{2}\sum_{l=1}^{N} \hbar\Omega_l (X - x_l)\left\{ \mathrm{e}^{\mathrm{i}\omega_l X/v} S_l^- + \mathrm{e}^{-\mathrm{i}\omega_l X/v} S_l^+ \right\}, \tag{2.5}$$

where X is the position operator of the particle and the l-th detector spin is located at the position x_l.

This model has a characteristic restriction in that the interaction occurs only when the particle spin is up. This is explicitly shown by the operator $\mathcal{P}_+ = \frac{1}{2} + I^z$ in the second term of (2.1).

We are already able to determine the time evolution of the density matrices starting from the initial density matrix of the form

$$\begin{aligned} W(0) &= |\Psi(0)\rangle\rangle\langle\langle\Psi(0)| \\ &= |I\rangle\langle I| \otimes |\,\psi\rangle\langle\psi| \otimes |\{\, \boldsymbol{z}_l^0\}\rangle\langle\{\boldsymbol{z}_l^0\}|\ , \end{aligned} \tag{2.6}$$

where

$$|I\rangle = a|+\rangle + b|-\rangle\ , |\psi\rangle = \int \mathrm{d}x \psi(x)|x\rangle, \tag{2.7}$$

and

$$|\{\boldsymbol{z}_l^0\}\rangle = \prod_{l=1}^{N} |\boldsymbol{z}_l^0\rangle. \tag{2.8}$$

In these expressions, $|I\rangle$ and $|\psi\rangle$ designate the spin state and the orbital state for the incident particle, respectively, and $|\boldsymbol{z}_l^0\rangle$ indicates the initial spin state for the detector in the spin coherent state representation, which is characterized by the vector $\boldsymbol{z}_l^0$ (the superscript 0 designates the initial state),

$$\boldsymbol{z}_l^0 = \begin{pmatrix} z_{l,+}^0 \\ z_{l,-}^0 \end{pmatrix} = z_l^0 \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi_l^0} \cos(\theta_l^0/2) \\ \mathrm{e}^{\mathrm{i}\phi_l^0} \sin(\theta_l^0/2) \end{pmatrix}, \qquad (2.9)$$

where z_l^0, θ_l^0 and ϕ_l^0 are a complex number, polar angle and azimuthal angle, respectively.

Thus, the time evolution of the density matrix is determined to be

$$W(t) \equiv \mathrm{e}^{-\mathrm{i}\mathcal{H}t/\hbar} W(0) \mathrm{e}^{\mathrm{i}\mathcal{H}t/\hbar} \qquad (2.10)$$

$$= \int \mathrm{d}x \int \mathrm{d}x' \psi(x)\psi^*(x')|x+vt\rangle\langle x'+vt|$$
$$\otimes\{|a|^2|+\rangle\langle+| \otimes |\{\boldsymbol{z}_l^{0(int)}(x,t)\}\rangle\langle\{\boldsymbol{z}_l^{0(int)}(x',t)\}| + |b|^2|-\rangle\langle-| \otimes |\{\boldsymbol{z}_l^{0(non)}\}\rangle\langle\{\boldsymbol{z}_l^{0(non)}\}|$$
$$+ab^*\mathrm{e}^{-\mathrm{i}\omega_I t}|+\rangle\langle-| \otimes |\{\boldsymbol{z}_l^{0(int)}(x,t)\}\rangle\langle\{\boldsymbol{z}_l^{0(non)}\}| + a^*b\mathrm{e}^{\mathrm{i}\omega_I t}|-\rangle\langle+| \otimes |\{\boldsymbol{z}_l^{0(non)}\}\rangle\langle\{\boldsymbol{z}_l^{0(int)}(x',t)\}|\}, \qquad (2.11)$$

where

$$\boldsymbol{z}_l^{0(int)}(x,t) = \begin{pmatrix} \left(z_{l,+}^0 \cos\dfrac{\Theta_l(x;t)}{2} - \mathrm{i}z_{l,-}^0 \mathrm{e}^{-\mathrm{i}\omega_l x/v} \sin\dfrac{\Theta_l(x;t)}{2}\right)\mathrm{e}^{-\mathrm{i}\omega_l t/2} \\ \left(z_{l,-}^0 \cos\dfrac{\Theta_l(x;t)}{2} - \mathrm{i}z_{l,+}^0 \mathrm{e}^{\mathrm{i}\omega_l x/v} \sin\dfrac{\Theta_l(x;t)}{2}\right)\mathrm{e}^{\mathrm{i}\omega_l t/2} \end{pmatrix} \qquad (2.12)$$

and

$$\boldsymbol{z}_l^{0(non)} = \begin{pmatrix} z_{l,+}^0 \mathrm{e}^{-\mathrm{i}\omega_l t/2} \\ z_{l,-}^0 \mathrm{e}^{\mathrm{i}\omega_l t/2} \end{pmatrix}. \qquad (2.13)$$

In the above expression, the quantity $\Theta_l(x;t)$ measures an effect due to the interaction up to time t:

$$\Theta_l(x;t) = \int_0^t \mathrm{d}t' \Omega_l(x+vt'-x_l), \qquad (2.14)$$

$\Omega_l(x)$ being defined by eq. (2.5).

Moreover, the reduced density matrix for the incident spin is calculated exactly by eliminating the irrelevant information:

$$\rho_I(t) = |a|^2|+\rangle\langle+| + |b|^2|-\rangle\langle-|$$
$$+ \int \mathrm{d}x|\psi(x)|^2 \left\{ ab^*\mathrm{e}^{-\mathrm{i}\omega_I t} \prod_{l=1}^N \left(\cos\frac{\Theta_l(x;t)}{2} \right.\right.$$
$$\left. -\mathrm{i}\sin\theta_l^0 \cos(\phi_l^0 - \omega_l x/v) \sin\frac{\Theta_l(x;t)}{2} \right)^{2S_l} |+\rangle\langle-|$$
$$\left. +c.c. \right\}. \qquad (2.15)$$

where S_l is the magnitude of l-th spin in the detector.

Similarly, we have for the detector,

$$\rho_D(t)$$
$$= \int \mathrm{d}x|\psi(x)|^2 \{ |a|^2|\{\boldsymbol{z}_l^{0(int)}(x,t)\}\rangle\langle\{\boldsymbol{z}_l^{0(int)}(x,t)\}|$$
$$+ |b|^2|\{\boldsymbol{z}_l^{0(non)}\}\rangle\langle\{\boldsymbol{z}_l^{0(non)}\}| \}, \qquad (2.16)$$

and for the l-th spin of the detector,

$$\rho_{l,D}(t) = \int \mathrm{d}x|\psi(x)|^2 \{ |a|^2|\boldsymbol{z}_l^{0(int)}(x,t)\rangle\langle\boldsymbol{z}_l^{0(int)}(x,t)|$$
$$+|b|^2|\boldsymbol{z}_l^{0(non)}\rangle\langle\boldsymbol{z}_l^{0(non)}| \}. \qquad (2.17)$$

Using these results, we can obtain the averages and the quasi-probability densities for the spin variables. Details are shown in the previous papers.[4-6] Here we quote the final expressions for the averages:

$$\langle I^-(t)\rangle = ab^*\mathrm{e}^{-\mathrm{i}\omega_I t} \int \mathrm{d}x|\psi(x)|^2 \prod_{l=1}^N \left[\cos\frac{\Theta_l(x;t)}{2} \right.$$
$$\left. -\mathrm{i}\sin\theta_l^0 \cos(\phi_l^0 - \omega_l x/v) \sin\frac{\Theta_l(x;t)}{2} \right]^{2S_l}, \qquad (2.18)$$

$$\langle I^z(t)\rangle = \frac{1}{2}\left(|a|^2 - |b|^2\right), \qquad (2.19)$$

$$\langle S_l^-(t)\rangle = S_l \mathrm{e}^{-\mathrm{i}\omega_l t} \left\{ \int \mathrm{d}x|\psi(x)|^2 \right.$$
$$\times|a|^2 \left[\left(\mathrm{e}^{-\mathrm{i}\phi_l^0} \cos^2\frac{\Theta_l(x;t)}{2} \right.\right.$$
$$\left. + \mathrm{e}^{\mathrm{i}(\phi_l^0 - 2\omega_l x/v)} \sin^2\frac{\Theta_l(x;t)}{2} \right) \sin\theta_l^0$$
$$\left. +\mathrm{i}\mathrm{e}^{-\mathrm{i}\omega_l x/v} \sin\Theta_l(x;t) \cos\theta_l^0 \right]$$
$$\left. + |b|^2 \mathrm{e}^{-\mathrm{i}\phi_l^0} \sin\theta_l^0 \right\}, \qquad (2.20)$$

and

$$\langle S_l^z(t)\rangle = S_l \left\{ |a|^2 \int \mathrm{d}x|\psi(x)|^2 (\cos\theta_l^0 \cos\Theta_l(x;t) \right.$$
$$+ \sin\theta_l^0 \sin(\phi_l^0 - \omega_l x/v) \sin\Theta_l(x;t))$$
$$\left. +|b|^2 \cos\theta_l^0 \right\}. \qquad (2.21)$$

These will be used in the subsequent sections.

§3. Time Evolution of Entropy

The purpose of this paper is to determine the entropy associated with the Coleman-Hepp model and to study the dynamics causing the decoherence.

We first note that the entropy of the quantum system is defined by

$$\mathcal{S} = -k_\mathrm{B} \mathrm{Tr}\rho \ln \rho, \qquad (3.1)$$

where ρ is the density matrix and k_B, the Boltzmann constant. To treat the spin 1/2 system, we consider the eigenvalue problem of ρ:

$$\rho|p_j\rangle = p_j|p_j\rangle. \tag{3.2}$$

The eigenvalues are solved to give

$$p_j = \frac{1}{2} \pm r \tag{3.3}$$

and thus the entropy $\mathcal{S}$ can be written as

$$\mathcal{S} = -k_\mathrm{B}\left\{ \left(\frac{1}{2}+r\right)\ln\left(\frac{1}{2}+r\right) + \left(\frac{1}{2}-r\right)\ln\left(\frac{1}{2}-r\right) \right\}, \tag{3.4}$$

where

$$r = \sqrt{\langle I^x(t)\rangle^2 + \langle I^y(t)\rangle^2 + \langle I^z(t)\rangle^2}, \tag{3.5}$$

for the particle spin. Then, the corresponding entropy for the l-th detector spin is obtained simply by replacing $\boldsymbol{I}$ by $\boldsymbol{S}_l$ in (3.5) when $S_l = 1/2$.

As mentioned above, the averages of the Coleman-Hepp model are already known, and therefore we can apply this formulation directly to the particle spin and the detector spin under the assumption that:

(i) In the initial state, a particle is located around $x = 0$ with the width Δ, whose spin is along the x-axis,

(ii) All the detector spins have the same magnitude and form a line along the z-axis with the same interval d.

(iii) The quantity $\Omega_l(x - x_l)$ localized around x_l with the width δ, explicitly, is given by

$$\Omega_l(x - x_l) = \frac{\Omega_l \cdot d}{\sqrt{2\pi\delta^2}} \mathrm{e}^{-(x-x_l)^2/2\delta^2}. \tag{3.6}$$

(iv) It is assumed that $\delta \gg \Delta$, giving $|\psi(x)|^2 = \delta(x)$. These assumptions yield the following:

$$\langle I^-(t)\rangle = \frac{1}{2}\mathrm{e}^{-\mathrm{i}\omega_I t}\prod_{l=1}^{N}\left[\cos\frac{\Theta_l(vt)}{2}\right]^{2S_l}, \tag{3.7}$$

$$\langle I^z(t)\rangle = 0, \tag{3.8}$$

and

$$\langle S_l^-(t)\rangle = -\frac{\mathrm{i}}{2}S_l \mathrm{e}^{-\mathrm{i}\omega_l t}\sin\Theta_l(vt), \tag{3.9}$$

and

$$\langle S_l^z(t)\rangle = -\frac{1}{2}S_l\left(\cos\Theta_l(vt) + 1\right), \tag{3.10}$$

where

$$\Theta_l(x) = \frac{1}{v}\int_{-\infty}^{x} \mathrm{d}x'\Omega_l(x' - x_l). \tag{3.11}$$

Therefore the time evolution of the entropy for the particle spin and the detector spins can be determined easily by substituting these expressions into eqs. (3.4) and (3.5) to the extent that we treat only the magnitude 1/2 spins. When the magnitude of the spin is arbitrary, we need to generalize the treatment; this will be done in a forthcoming paper.

§4. Numerical Studies and Concluding Remarks

We can use the entropy as a measure of disorder on the spin state. In the spin 1/2 system, $\mathcal{S}/k_\mathrm{B}$ ranges between 0 and $\ln 2$, which correspond to the pure state and the randomized state, respectively. On these grounds, we reveal several consequences derived from (3.7)–(3.10). In Fig. 1 we show the time evolution of the entropy for the particle spin with the parameter $\Omega_l d/v = 5$ and $N = 10$. At $t = 0$, $\mathcal{S}/k_\mathrm{B}$ has the value 0 and after passing through the detector (i.e., after $\hat{t}(\equiv vt/d) = 10$), $\mathcal{S}/k_\mathrm{B}$ becomes almost $\ln 2$. The oscillation occurs in the detector, due to the strong interaction.

Next we treat somewhat different situations in Figs. 2 and 3. It is already shown that the decoherence (dephasing) phenomena occur when the parameters S_l and/or N become very large. Among others, the dephasing phenomena were observed even for $N = 1$ when S_l is large. This case is treated in Figs. 2 and 3. In Fig. 2, we see that the entropy increases monotonically when the interaction is weak ($\Omega_l d/v = 1$). However, as $\Omega_l d/v$ increases, the entropy oscillates and approaches a constant value. The stronger the interaction, the more oscillations appear. In Fig. 3, the two different cases, $S_l = 1/2$ and $S_l = 20$, are treated when $\Omega_l d/v = 8$. They behave essentially the same. However, with increasing S_l, the entropy change becomes very sharp and the final value

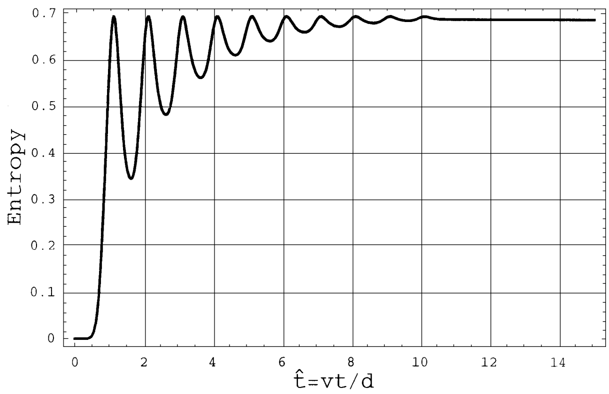

Fig. 1. Time evolution of the entropy, $\mathcal{S}/k_\mathrm{B}$, for the particle spin as a function of $\hat{t} \equiv vt/d$ for the parameters $\Omega_l d/v = 5$, $N = 10$, $\delta/d = 0.25$ and $S_l = 1/2$.

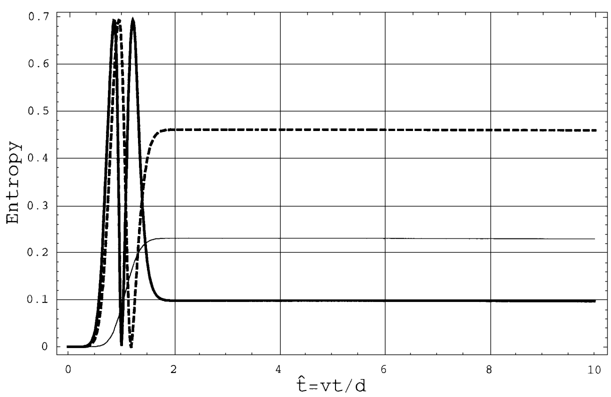

Fig. 2. Time evolution of $\mathcal{S}/k_\mathrm{B}$ for the particle spin as a function of $\hat{t}$ by changing $\Omega_l d/v$ for $N = 1$, $\delta/d = 0.25$ and $S_l = 1/2$. The thin solid, broken, and thick solid lines correspond to $\Omega_l d/v = 1$, 8, and 12, respectively.

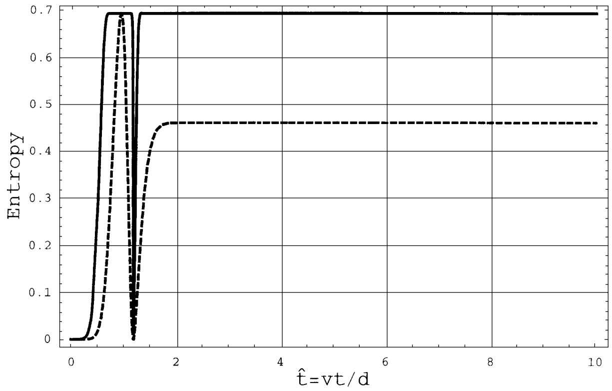

Fig. 3. Time evolution of $\mathcal{S}/k_{\rm B}$ for the particle spin as a function of $\hat{t}$ by changing S_l for $N=1$, $\delta/d=0.25$ and $\Omega_l d/v=8$. The solid and broken lines correspond to $S_l=20$ and $S_l=1/2$, respectively.

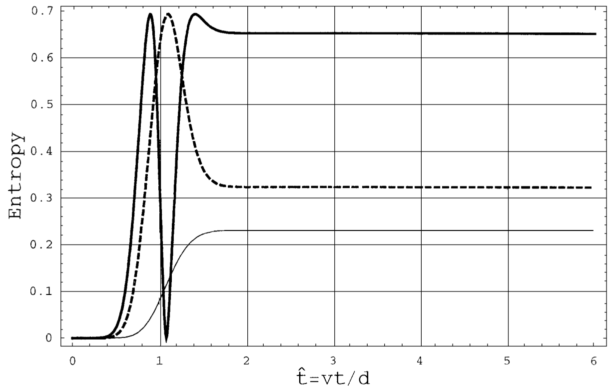

Fig. 4. Time evolution of $\mathcal{S}/k_{\rm B}$ for the first detector spin as a function of $\hat{t}$ by changing $\Omega_l d/v$ for parameters $\delta/d=0.25$, $S_1=1/2$. The thin solid, broken, and thick solid lines correspond to $\Omega_l d/v=1$, 5, and 10, respectively.

of $\mathcal{S}/k_{\rm B}$ approaches $\ln 2$. Further increase in S_l results in vanishing of the oscillation. Thus, we find that the interaction strength $\Omega_l d/v$ and the apparatus parameter S_l play different roles in the decoherence (dephasing) processes.

We show in Fig. 4 the time evolution of the entropy for the first detector spin ($l=1$). When the interaction strength Ω_l is changed ($\Omega_l d/v=1$, 5, 10), the resulting behavior is similar to that found in Fig. 2. Finally, we discuss the total entropy of the detector. Each constituent spin in the detector results in the same behavior as in Fig. 4. The total entropy is obtained as a sum of the individual entropies. We may be observing several typical behaviors: one an oscillating increase and the other a stepwise increase in entropy as a function of time.

In conclusion, we were able to exactly obtain the quantum mechanical entropy of the generalized Coleman-Hepp model. Namely, we could explicitly calculate the entropy of spin freedom for the incident particle and the detector. Further details will be reported elsewhere in relation to the Wehrl[10]-Lieb[11] entropy.

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