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Hirota's Method and the Singular Manifold Expansion

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A system of equations $u_t+(u^2/2+\alpha u_{mx}+\beta u_{nx})_x=0$ (m, n: positive integers, $\beta\neq 0$) is studied by means of Hirota's method and the singular manifold expansion. The singular manifold expansion yields the transformation of the system into bilinear forms or higher order ones and we obtain some explicit solutions of the system in physically interesting but non-integrable cases.

§1. Introduction

A unique method is developed by Hirota[1)] to find explicit solutions of nonlinear partial differential equations (p.d.e.'s). The method applies not only to integrable p.d.e.'s but also to non-integrable ones. In the integrable case, N-soliton solutions are obtained by means of the method, and some exact solutions are found even in the non-integrable case. A crucial procedure of the method is to make the appropriate transformation of the p.d.e. concerned into bilinear forms. However, there are no systematic ways to find the transformation.

Weiss *et al.* employed the singular manifold expansion to examine the Painleve property and derived the Bäcklund transformation by truncating the expansion.[2)] The relation between the singular manifold and the Hirota's function in bilinear forms was established in the integrable case by Gibbon *et al.*[3)]

In this paper, we derive the transformation of the following system of generally non-integrable p.d.e.'s:

$$u_t+(u^2/2+\alpha u_{mx}+\beta u_{nx})_x=0, \qquad (1)$$

into bilinear forms or higher order ones, where m, n are integers ($m<n$) and α, β are constant ($\beta\neq 0$). The transformation is introduced by the truncation of the singular manifold expansion at a different level from that of Weiss. The system (1) includes both integrable p.d.e.'s (the K-dV and Burgers equations) and non-integrable p.d.e.'s such as the K-dV-Burgers, Kuramoto[4)] and fifth-order dispersive K-dV equations. In the next section, the integrable case is discussed as an example. In §3, the transformation of non-integrable p.d.e.'s is derived and some explicit solutions are given.

§2. Integrable Case

In this section, we consider the case in which $\alpha=0$, $n=2$ (K-dV) and $\alpha=0$, $n=1$ (Burgers) and recover Hirota's transformation[1)] and the Cole-Hopf transformation, respectively. The singular manifold expansion is introduced by assuming that u can be expanded as

$$u=\phi^{-r}\sum_{j=0}^{\infty}u_j\phi^j, \qquad (2)$$

where u_j and ϕ are functions of independent variables and ϕ determines a singular manifold.[2)] On substitution of eq. (2) into eq. (1), the leading order is easily seen to be $r=2$, $u_0=-12\beta\phi_x^2$ for the K-dV equation and $r=1$, $u_0=2\beta\phi_x$ for the Burgers equation. The higher order terms yield u_j's ($j\geqq 1$) recursively except at resonances. Let us truncate the expansion so that $u_j=0$ for $j>r-1$, then we have $u_1=12\beta\phi_{2x}$, $u_j=0$ ($j>1$) for the K-dV equation and $u_j=0$ ($j>0$) for the Burgers equation. Thus, we have Hirota's transformation $u=12\beta(\ln\phi)_{2x}$ and the Cole-Hopf transformation $u=2\beta(\ln\phi)_x$. In order that the truncation be valid, the following conditions are respectively required:

$$\{D_xD_t+\beta D_x{}^4\}(\phi\cdot\phi)=0 \quad \text{(K-dV)}, \qquad (3)$$

$$\phi_t+\beta\phi_{2x}=0 \quad \text{(Burgers)},$$

where eq. (3) is the well-known Hirota's bilinear form of the K-dV equation and D_x, D_t are Hirota's bilinear operators defined by

$$D_x^n(\phi\cdot\phi)=\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^n \phi(x)\phi(x')\Big]_{x=x'}.$$

It should be noted that Weiss *et al.* truncate the expansion such that $u_j=0$ $(j>r)$, and obtain the Bäcklund transformation.[2)]

§3. Non-Integrable Case

Even in the non-integrable case, we apply the same rule of truncation as that in the integrable case, that is, $u_j=0$ for $j>r-1$, and obtain useful transformations.

(1) K-dV-Burgers equation:

$$u_t+(u^2/2+\alpha u_x+\beta u_{2x})_x=0.$$

The leading order is seen to be $r=2$, $u_0=-12\beta\phi_x^2$ and the next order terms give $u_1=12\beta\phi_{2x}+(12\alpha/5)\phi_x$. By setting $u_j=0$ for $j>1$, we have

$$u=12\beta(\ln\phi)_{2x}+(12\alpha/5)(\ln\phi)_x, \tag{4}$$

which is the linear combination of Hirota's transformation and the Cole-Hopf transformation. As the condition necessary for the truncation, we obtain

$$\{D_xD_t+\beta D_x^4\}(\phi\cdot\phi)-(\alpha^2/25\beta)D_x^2(\phi\cdot\phi)+(12\alpha/5)D_x(\phi_{2x}\cdot\phi)=-(2\alpha/5\beta)\phi(\phi_t+6\alpha\phi_{2x}/5). \tag{5}$$

It is easy to see that $\phi=1+\exp(kx+\omega t)$ is a solution of eq. (5), where $\omega=-6\alpha k^2/5$ and $k=\pm\alpha/(5\beta)$. Using the transformation (4), we have a solution of the K-dV-Burgers equation in the form of superposition of a soliton and a shock:

$$u=3\beta k^2\,\mathrm{sech}^2\{(kx+\omega t)/2\}+(12\alpha/5)k\exp(kx+\omega t)/\{1+\exp(kx+\omega t)\}.$$

(2) Kuramoto equation:

$$u_t+(u^2/2+\alpha u_x-\beta u_{3x})_x=0.$$

In this case, we have $r=3$, $u_0=-120\beta\phi_x^3$, $u_1=180\beta\phi_x\phi_{2x}$ and $u_2=-60\beta\phi_{3x}+60\alpha\phi_x/19$. According to the truncation rule, we set $u_j=0$ $(j>2)$ and have a transformation

$$u=60\beta(\ln\phi)_{3x}+60\alpha(\ln\phi)_x/19. \tag{6}$$

Then, the Kuramoto equation is transformed into the higher order form

$$\begin{aligned}&\{\alpha\phi^2/19-\beta D_x^2(\phi\cdot\phi)\}(\phi_t+30\alpha\phi_{2x}/19)+\phi[\beta D_x^2\{(\phi_t+30\alpha\phi_{2x}/19)\cdot\phi\}-11(\alpha/19)^2D_x^2(\phi\cdot\phi)\\&\quad-10\alpha\beta D_x^4(\phi\cdot\phi)/19-30\beta^2D_x^2(\phi_{2x}\cdot\phi_{2x})+\beta^2D_x^6(\phi\cdot\phi)]+30\beta^2D_x\{D_x^2(\phi_x\cdot\phi_x)\cdot\phi_x\}=0,\end{aligned} \tag{7}$$

which may be called a trilinear form. The two explicit solutions of eq. (7) are obtained as

$$\phi=1+\exp(kx+\omega t),\quad \omega=-30\alpha k^2/19,\quad k^2=-\alpha/(19\beta)\quad\text{or}\quad k^2=11\alpha/(19\beta), \tag{8}$$

and

$$\begin{aligned}&\phi=E_++E_-,\quad E_\pm=\exp(\pm kx+\omega t),\quad \omega=-30\alpha k^2/19,\\&(2k)^2=-\alpha/(19\beta)\quad\text{or}\quad(2k)^2=11\alpha/(19\beta).\end{aligned} \tag{9}$$

From eqs. (6) and (8), we have a propagating shock solution

$$u=(60/19)\alpha k\exp(kx+\omega t)/\{1+\exp(kx+\omega t)\}+15\beta k^3\tanh\{(kx+\omega t)/2\}\,\mathrm{sech}^2\{(kx+\omega t)/2\}.$$

The colliding shock solution found by Kuramoto *et al.*[4)] is recovered from eqs. (6) and (9):

$$u=(60/19)\alpha k\tanh kx-120\beta k^3\,\mathrm{sech}^2kx\tanh kx.$$

(3) K-dV equation with the fifth order dispersion:

$$u_t+(u^2/2+\alpha u_{2x}+\beta u_{4x})_x=0.$$

Following the same procedure as given above, we obtain a transformation

$$u=280\{\beta(\ln\phi)_{4x}+\alpha(\ln\phi)_{2x}/13\}, \tag{10}$$

and a tetralinear form

$$\phi^2\{D_xD_t+D_x^4(\beta D_x^2+\alpha)\}\{\beta D_x^2+\alpha/13\}(\phi\cdot\phi)-3\beta D_x^2(\phi\cdot\phi)D_xD_t(\phi\cdot\phi)+\beta^2[35\{D_x^4(\phi\cdot\phi)\}^2 -28D_x^2(\phi\cdot\phi)D_x^6(\phi\cdot\phi)]-70\alpha\beta/13D_x^2(\phi\cdot\phi)D_x^4(\phi\cdot\phi)+31(\alpha/13)^2\{D_x^2(\phi\cdot\phi)\}^2=0. \quad (11)$$

When $\alpha\beta<0$, it is found that eq. (11) has a solution $\phi=1+\exp(kx+\omega t)$, $\omega=-36\beta k^5$ and $k^2=-\alpha/(13\beta)$. The transformation (10) gives

$$u=-105(\alpha/13)^2/\beta\,\mathrm{sech}^4\{(kx+\omega t)/2\},$$

which is obtained by means of a different method.[6]

§4. Concluding Remarks

The transformation obtained in previous sections is easily generalized for arbitrary m and n in eq. (1):

$$u=a(\ln\phi)_{nx}+b(\ln\phi)_{mx},$$

where a and b are constant. Then, eq. (1) is transformed into the n-th order homogeneous equation with respect to ϕ. We can also consider the extended version of eq. (1)

$$u_t+(u^2/2+\alpha u_{lx}+\beta u_{mx}+\gamma u_{nx})_x=0. \quad (12)$$

In the case that $l=1$, $m=2$ and $n=3$,[5] for example, the singular manifold expansion yields the transformation

$$u=60\gamma(\ln\phi)_{3x}+15\beta(\ln\phi)_{2x} +60(\alpha-\beta^2/16\gamma)/19(\ln\phi)_x,$$

by which eq. (12) is transformed into the similar trilinear form as eq. (7). However, no explicit solutions of the trilinear form are known yet.

Since eq. (1) is invariant for the Galilei transformation

$$x'=x-vt,\quad t'=t,\quad u'=u+v,$$

a constant speed (v) is added to our solutions.

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